

## UNIT I – INTRODUCTION

### WEIGHTED RESIDUAL METHODS

General Procedures,  $\int_D (\delta(x - x_i) R(x; a_1, a_2, a_3, \dots, a_n) dx = 0$

Where  $w_i =$  Weight function,  $D =$  Domain,  $R =$  Residual

#### *a. Point Collocation Method:*

Residual (R) = 0

#### *b. Subdomain Collocation Method:*

$$\int R dx = 0$$

#### *c. Least Squares Method:*

$$\int_D R^2 dx = \text{minimum}$$

$$\partial I / \partial a = \int R (\partial R / \partial a) dx = 0$$

#### *d. Galerkin's Method:*

$$\int_D W_i R dx = 0$$

### RITZ METHOD (Variational Approach)

Total Potential = Internal Potential Energy - External Potential Energy  
= Strain Energy – Work done

Total Potential ( $\pi$ ) = U - W

### For simply Supported Beam with Uniformly Distributed Load:

$$\text{Strain energy (U)} = \frac{EI}{2} \int_0^l \left( \frac{d^2 y}{dx^2} \right)^2 dx$$

$$\text{Workdone (W)} = \int_0^l w \cdot y dx$$

Where,  $y = a_1 \sin \pi x / l + a_2 \sin 3 \pi x / l$

## For simply Supported Beam with Point Load:

$$\text{Strain energy (U)} = \frac{EI}{2} \int_0^l \left( \frac{d^2y}{dx^2} \right)^2 dx$$

$$\text{Workdone (w)} = P \cdot y_{\max}$$

$$\text{Where, } y = a_1 \sin \pi x / l + a_2 \sin 3 \pi x / l$$

## For Cantilever Beam with Uniformly Distributed Load:

$$\text{Strain energy (U)} = \frac{EI}{2} \int_0^l \left( \frac{d^2y}{dx^2} \right)^2 dx$$

$$\text{Workdone (W)} = \int_0^l w \cdot y \, dx$$

$$\text{Where, } y = A(1 - \cos \pi x / 2l)$$

## For Cantilever Beam with Point Load:

$$\text{Strain energy (U)} = \frac{EI}{2} \int_0^l \left( \frac{d^2y}{dx^2} \right)^2 dx$$

$$\text{Workdone (W)} = P \cdot y_{\max}$$

$$\text{Where, } y = A(1 - \cos \pi x / 2l)$$

## UNIT II – ONE DIMENSIONAL (1D) ELEMENTS

### 01. Stress – Strain relationship,

$$\text{Stress, } \sigma \text{ (N/mm}^2\text{)} = \text{Young's Modulus, } E \text{ (N/mm}^2\text{)} \times \text{Strain, } e$$

### 02. Strain – Displacement Relationship

$$\text{Strain } \{e\} = du / dx$$

### 03. Strain $\{e\} = [B] \{U\}$

Where,

$$\begin{aligned} \{e\} &= \text{Strain Matrix} \\ [B] &= \text{Strain – Displacement Matrix} \\ \{U\} &= \text{Degree of Freedom (Displacement)} \end{aligned}$$

### 04. Stress $\{\sigma\} = [E] \{e\} = [D] \{e\} = [D] [B] \{u\}$

Where,  $[E] = [D] = \text{Stress – Strain Matrix}$

### 05. General Equation for Stiffness Matrix, $[K] = \int_v [B]^T [D] [B] dv$

Where,

$$\begin{aligned} [B] &= \text{Strain - Displacement relationship Matrix} \\ [D] &= \text{Stress - Strain relationship Matrix} \end{aligned}$$

### 06. General Equation for Force Vector, $\{F\} = [K] \{U\}$

Where,

$$\begin{aligned} \{F\} &= \text{Global Force Vector} \\ [K] &= \text{Stiffness Matrix} \\ \{U\} &= \text{Global Displacements} \end{aligned}$$

### (i) 1D BAR ELEMENTS:

### 01. Stiffness Matrix for 1D Bar element, $[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

### 02. Force Vector for 2 noded 1D Bar element, $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

### 03. If self weight is considered, the Load / Force Vector, $\{F\}_E = \frac{\rho A l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

where  $\rho = \text{Density, N/mm}^3$

**(ii) 1D TRUSS ELEMENTS:**

01. General Equation for Stiffness Matrix,  $[K] = \frac{A_e E_e}{l_e} \begin{pmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{pmatrix}$

02. Force vector for 2-noded Truss elements,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \frac{A_e E_e}{l_e} \begin{pmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Where,  $l = \cos \theta = (x_2 - x_1) / l_e$

$m = \sin \theta = (y_2 - y_1) / l_e$

Length of the element ( $l_e$ ) =  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

**(iii) 1D BEAM ELEMENTS:**

01. Force Vector for Two noded Beam Element

$$\begin{Bmatrix} F_1 \\ m_1 \\ F_2 \\ m_2 \end{Bmatrix} = \frac{E_e I_e}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix} \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \end{Bmatrix}$$

02. Stiffness Matrix for Two noded Beam Element

$$[K] = \frac{E_e I_e}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

Where,

I = Moment of Inertia (mm<sup>4</sup>)

L = Length of the Beam (mm)



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06. Stiffness Matrix for 2D element / CST Element,  $[K] = [B]^T [D] [B] A t$

Where,  $A =$  Area of the triangular element,  $mm^2 = \frac{1}{2} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$

$t =$  Thickness of the triangular (CST) element, mm

07. Maximum Normal Stress,  $\sigma_{max} = \sigma_1 = \left( \frac{\sigma_x + \sigma_y}{2} \right) + \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$

08. Minimum Normal Stress,  $\sigma_{min} = \sigma_2 = \left( \frac{\sigma_x + \sigma_y}{2} \right) - \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$

09. Principal angle,  $\tan 2\theta_p = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$

10. Stress equation for Axisymmetric Element,

Stress  $\{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$       Where,  $\sigma_r =$  Radial Stress  
 $\sigma_\theta =$  Longitudinal Stress  
 $\sigma_z =$  Circumferential Stress  
 $\tau_{rz} =$  Shear Stress

Strain,  $\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$       Where,  $e_r =$  Radial Stress  
 $e_\theta =$  Longitudinal Stress  
 $e_z =$  Circumferential Stress  
 $\gamma_{rz} =$  Shear Stress

11. Stiffness matrix for Two Dimensional Axisymmetric Problems,

$[K] = 2 \pi r A [B]^T [D] [B]$

Where, Co-ordinate  $r = r_1 + r_2 + r_3 / 3$       &       $z = z_1 + z_2 + z_3 / 3$

$$A = \text{Area of the Triangle} = \frac{1}{2} \begin{pmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{pmatrix}$$

**12. Strain – Displacement Relationship Matrix for Axisymmetric elements,**

$$[B] = \begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 + \gamma_1 z}{r} & \frac{\alpha_1 + \beta_1 + \gamma_1 z}{r} & 0 & 0 & \frac{\alpha_3 + \beta_3 + \gamma_3 z}{r} & \frac{\alpha_3 + \beta_3 + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix}$$

Where,

$$\begin{aligned} \alpha_1 &= r_2 z_3 - r_3 z_2 & \beta_1 &= z_2 - z_3 & \gamma_1 &= r_3 - r_2 \\ \alpha_2 &= r_3 z_1 - r_1 z_3 & \beta_2 &= z_3 - z_1 & \gamma_2 &= r_1 - r_3 \\ \alpha_3 &= r_1 z_2 - r_2 z_1 & \beta_3 &= z_1 - z_2 & \gamma_3 &= r_2 - r_1 \end{aligned}$$

**13. Stress – Strain Relationship Matrix [D] for Axisymmetric Triangular elements,**

$$[D] = \frac{E}{(1 + \mu)(1 - \mu^2)} \begin{pmatrix} (1 - \mu) & \mu & \mu & 0 \\ \mu & (1 - \mu) & \mu & 0 \\ \mu & \mu & (1 - \mu) & 0 \\ 0 & 0 & 0 & \frac{1 - 2\mu}{2} \end{pmatrix}$$

Where,  $\mu$  = Poisson's Ratio  
 $E$  = Young's Modulus

## UNIT IV – HEAT TRANSFER APPLICATIONS

### TEMPERATURE PROBLEMS FOR 1D ELEMENTS

$$01. \text{ Temperature Force } \{F\} = E A \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Where,  
E = young's Modulus (N/mm<sup>2</sup>)  
A = Area of the Element (mm<sup>2</sup>)  
 $\alpha$  = Coefficient of thermal expansion (°C)  
 $\Delta T$  = Temperature Difference (°C)

$$02. \text{ Thermal stress } \{\sigma\} = E (du/dx) - E \alpha \Delta T \quad \text{Where } du / dx = u_1 - u_2 / l$$

### TEMPERATURE PROBLEMS FOR 2D PROBLEMS ELEMENTS

$$01. \text{ Temperature Force, } \{\theta\} \text{ or } \{f\} = [B]^T [D] \{e_0\} t A$$

a. For Plane Stress Problems,

$$\text{Initial Strain } \{e_0\} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

b. For Plane Strain Problems,

$$\text{Initial Strain } \{e_0\} = (1 + \nu) \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

### TEMPERATURE PROBLEMS AXISYMMETRIC TRIANGULAR ELEMENTS

Temperature Effects:

$$\text{For Axisymmetric Triangular elements, Temperature Force, } \{f\}_t = [B]^T [D] \{e\}_t * 2 \pi r A$$

Where,

$$\{f\}_t = \begin{Bmatrix} F_{1u} \\ F_{1w} \\ F_{2u} \\ F_{2w} \\ F_{3u} \\ F_{3w} \end{Bmatrix} \quad \text{Strain } \{e\} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \\ \alpha \Delta T \end{Bmatrix}$$

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## HEAT TRANSFER PROBLEMS FOR 1D ELEMENTS:

a. General equation for Force Vector,  $\{F\} = [K_C] \{T\}$

b. Stiffness Matrix for 1D Heat conduction Element,  $[K_C] = \frac{A k}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

c. For Heat convection Problems,

$$\left[ \begin{matrix} A k / l & & & \\ & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & + h A & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = h T A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Where,  $k$  = Thermal conductivity of element, W/mK

$A$  = Area of the element,  $m^2$

$l$  = Length of the element, mm

$h$  = Heat transfer Coefficient,  $W/m^2K$

$T$  = fluid Temperature, K

$T$  = Temperature, K

**UNIT V – HIGHER ORDER AND ISOPARAMETRIC ELEMENTS**

**01. Shape Function for 4 Noded Rectangular Elements (Using Natural Co-Ordinate)**

$$N_1 = \frac{1}{4} (1 - \xi) (1 - \eta) \qquad N_3 = \frac{1}{4} (1 + \xi) (1 + \eta)$$

$$N_2 = \frac{1}{4} (1 + \xi) (1 - \eta) \qquad N_4 = \frac{1}{4} (1 - \xi) (1 + \eta)$$

**02. Displacement,**

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 \quad \& \quad v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

**03. To find a point of P,**

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \quad \& \quad y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

**04. Jaccobian Matrix, [J]** = 
$$\begin{pmatrix} \partial x / \partial \xi & \partial y / \partial \xi \\ \partial x / \partial \eta & \partial y / \partial \eta \end{pmatrix}$$

Where,

$$J_{11} = \frac{1}{4} \left[ - (1 - \eta) x_1 + (1 - \eta) x_2 + (1 + \eta) x_3 - (1 + \eta) x_4 \right]$$

$$J_{12} = \frac{1}{4} \left[ - (1 - \eta) y_1 + (1 - \eta) y_2 + (1 + \eta) y_3 - (1 + \eta) y_4 \right]$$

$$\mathbf{J}_{21} = \frac{1}{4} \left[ - (1 - \epsilon) x_1 - (1 + \epsilon) x_2 + (1 + \epsilon) x_3 + (1 - \epsilon) x_4 \right]$$

$$\mathbf{J}_{22} = \frac{1}{4} \left[ - (1 - \epsilon) y_1 - (1 + \epsilon) y_2 + (1 + \epsilon) y_3 + (1 - \epsilon) y_4 \right]$$

05. Strain – Displacement Relationship Matrix for Isoparmetric elements,

$$[\mathbf{B}] = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \mathbf{J}_{22} & -\mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & -\mathbf{J}_{21} & \mathbf{J}_{11} \\ \mathbf{J}_{21} & \mathbf{J}_{11} & \mathbf{J}_{22} & -\mathbf{J}_{12} \end{bmatrix}$$

$$\times 1/4 \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) \end{bmatrix}$$

06. Stiffness matrix for quadrilateral element,  $[\mathbf{K}] = t \iint [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}] |\mathbf{J}| \partial x \partial y$

07. Stiffness matrix for natural co-ordinates,  $[\mathbf{K}] = t \iint [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}] |\mathbf{J}| \partial \epsilon \partial \eta$

08. Stress – Strain  $[\mathbf{D}]$  Relationship Matrix,

*a. FOR PLANE STRESS PROBLEM,*

$$[\mathbf{D}] = \frac{E}{(1 - \mu^2)} \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{pmatrix}$$

*b. FOR PLANE STRAIN PROBLEM,*

$$[\mathbf{D}] = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{pmatrix} (1 - \mu) & \mu & 0 \\ \mu & (1 - \mu) & 0 \\ 0 & 0 & \frac{1 - 2\mu}{2} \end{pmatrix}$$

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09. Element Force vector,  $\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$

Where ,  $N$  is the shape function for 4 noded Quadrilateral elements

## 10. Numerical Integration (Gaussian Quadrature)

Where  $w_i =$  Weight function

$F(x_i) =$  values of function at pre determined points

No. of points	Location, $x_i$	Corresponding weights, $w_i$
1	$x_1 = 0.000\dots$	2.000
2	$x_1 = + \sqrt{1/3} = + 0.577350269189$ $x_2 = - \sqrt{1/3} = - 0.577350269189$	1.0000
3	$x_1 = + \sqrt{3/5} = + 0.774596669241$ $x_3 = - \sqrt{3/5} = - 0.774596669241$ $x_2 = 0.0000$	$5/9 = 0.5555555555$ $5/9 = 0.5555555555$ $8/9 = 0.8888888888$
4	$x_1 = + 0.8611363116$ $x_4 = - 0.8611363116$ $x_2 = + 0.3399810436$ $x_3 = - 0.3399810436$	0.3478548451 0.3478548451 0.6521451549 0.6521451549